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HERMITE-CHEBYSHEV POLYNOMIALS WITH THEIR GENERALIZED FORM

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Abstract

The main purpose of this paper is to present Hermite-Chebyshev polynomials and to give some properties of Hermite and Chebyshev polynomials. We derive operational identities, generating functions, and integral representation for power series satisfied by Hermite, Chebyshev, and Hermite-Chebyshev polynomials. Furthermore, for these Hermite-Chebyshev polynomials, we give operational rules with operators, often exploited in the theory of exponential operators. Finally, some definitions of Hermite-Chebyshev polynomials also of two, three and in turn several index are derived and new families of polynomials.

1. Introduction and Preliminaries

Special functions appear in statistics, Lie group theory, and number theory. The Hermite polynomials of the associated generating functions is reformulated within the framework of an operational formalism [4, 5, 6, 7, 10, 12]. In the case of generalized special functions, the use of operational techniques, combined with the principle of monomiality [2, 3, 9] has provided new means of analysis for the derivation of the solution of large classes of partial differential equations often encountered in physical problems [11], offers a powerful tool to treat the relevant generating functions and the differential equations they satisfy. The results are interpreted in terms of single, several variables, single index, index two, three and in turn p-index in terms of Hermite polynomials defined by Srivastava [14, 15]. The reason of interest for this family of Hermite polynomials is due to their intrinsic mathematical importance and to the fact that these polynomials have applications in physics.

In this paper, Hermite-Chebyshev polynomials are introduced and studied. We calculate summations, integral representation, and derive raising operators for Hermite-Chebyshev polynomials and of its generalization to the Hermite-Chebyshev polynomials. Before entering into more technical details, we will introduce some identities that will be largely exploited in this work. Finally, the end in this paper of an attempt of unify several results in the theory of polynomials, also in Hermite polynomials of one or more variables, author has defined the multi-index Hermite-Chebyshev polynomials.

The Crofton operational rule defined by [8]

$$\exp\left(\alpha \frac{d^m}{dx^m}\right)[f(x)g(x)] = f\left(x + m\alpha \frac{d^{m-1}}{dx^{m-1}}\right)\exp\left(\alpha \frac{d^m}{dx^m}\right)g(x). \quad (1.1)$$

The Burchnall identity is defined by [1]

$$\exp\left(y \frac{d^m}{dx^m}\right)x^n = \left(x + my \frac{d^{m-1}}{dx^{m-1}}\right)^n. \quad (1.2)$$

The n-th Hermite polynomials are defined by the following the generating function:

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x) = \exp\left(2xt - t^2\right), \quad (1.3)$$

and the explicit form

$$H_n(x) = n! \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k (2x)^{n-2k}}{k!\,(n-2k)!}; \quad n \geq 0, \quad (1.4)$$

where $\lfloor a \rfloor$ is the standard floor function which maps a real number a to its next smallest integer. According to [13], the Hermite polynomials satisfy the generating function

$$\sum_{n=0}^{\infty} \frac{(\alpha)_n H_n(x) t^n}{n!} = (1-2xt)^{-\alpha} {}_2F_0\left(\frac{\alpha}{2}, \frac{\alpha+1}{2}; -; \frac{-4t^2}{(1-2xt)^2}\right). \quad (1.5)$$

The $H_n(x)$ is defined through the operational identity

$$H_n(x) = \exp\left(-\frac{1}{4}\frac{d^2}{dx^2}\right)(2x)^n, \quad (1.6)$$

and the inverse of (1.6) allows us to conclude that

$$(2x)^n = \exp\left(\frac{1}{4}\frac{d^2}{dx^2}\right)H_n(x). \quad (1.7)$$

The Hermite polynomials have simple and useful representations in terms of definite integrals containing the variable x as parameter

$$H_n(x) = \frac{(-i)^n 2^n}{\sqrt{\pi}} \exp\left(x^2\right) \int_{-\infty}^{\infty} \exp\left(-t^2 + 2ixt\right) t^n dt. \tag{1.8}$$

The Chebyshev polynomials are defined by

$$U_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k (n-k)!\,(2x)^{n-2k}}{k!\,(n-2k)!}; \quad n \geq 0, \tag{1.9}$$

which are specified by the generating function

$$\sum_{n=0}^{\infty} U_n(x) t^n = (1 - 2xt + t^2)^{-1}; \quad |x| \leq 1, \quad |t| < \infty. \tag{1.10}$$

2. On Hermite and Chebyshev Polynomials

This section gives some properties of Hermite and Chebyshev polynomials. We start with the following theorem:

Theorem 2.1. *For* $k \in \mathbb{N}$,

$$\sum_{s=0}^{\lfloor n/2 \rfloor} \frac{(-k)^s (2kx)^{n-2s}}{s!\,(n-2s)!} = \sum_{n_1 + n_2 + \ldots + n_k = n} \frac{H_{n_1}(x) H_{n_2}(x) \ldots H_{n_k}(x)}{n_1!\, n_2! \ldots n_k!}. \tag{2.1}$$

Proof. Let us consider

$$f(x\sqrt{k},\, t\sqrt{k}) = \exp(2kxt - kt^2) = \exp(2kxt) \exp(-kt^2).$$

Hence we have

$$f(x\sqrt{k},\, t\sqrt{k}) = \sum_{n=0}^{\infty} \frac{(2kxt)^n}{n!} \sum_{s=0}^{\infty} \frac{(-kt^2)^s}{s!}$$

$$= \sum_{n=0}^{\infty} \sum_{s=0}^{\lfloor n/2 \rfloor} \frac{(-k)^s (2kx)^{n-2s}}{s!\,(n-2s)!} t^n. \tag{2.2}$$

On the other hand, from (1.3), we get

$$\exp\left(2kxt - kt^2\right) = \left[\exp\left(2xt - t^2\right)\right]^k = \left[\sum_{n=0}^{\infty}\frac{t^n}{n!}H_n(x)\right]^k$$

$$= \sum_{n=0}^{\infty}\left[\sum_{n_1+n_2+\ldots+n_k=n}\frac{H_{n_1}(x)H_{n_2}(x)\ldots H_{n_k}(x)}{n_1!\,n_2!\ldots n_k!}\right]t^n. \quad (2.3)$$

Combining (2.2) and (2.3) gives (2.1). □

Theorem 2.2. *For* $k \in \mathbb{N}$,

$$\sum_{s=0}^{[\frac{1}{2}n]}\frac{(-k)^s(2x_1+2x_2+\ldots+2x_k)^{n-2s}}{s!\,(n-2s)!} = \sum_{n_1+n_2+\ldots+n_k=n}\frac{H_{n_1}(x_1)H_{n_2}(x_2)\ldots H_{n_k}(x_k)}{n_1!\,n_2!\ldots n_k!}.$$

(2.4)

Proof. Let $g(x_1, x_2, \ldots, x_k, t) = \exp\left(2(x_1+x_2+\ldots+x_k)t - t^2\right)$. For $g(\frac{x_1}{\sqrt{k}}, \frac{x_2}{\sqrt{k}}, \ldots, \frac{x_k}{\sqrt{k}}, t\sqrt{k})$, using the power series and taking $n-2s$ instead of, we can write

$$g(\frac{x_1}{\sqrt{k}}, \frac{x_2}{\sqrt{k}}, \ldots, \frac{x_k}{\sqrt{k}}, t\sqrt{k}) = \exp\left(2k(x_1+x_2+\ldots+x_k)t - kt^2\right)$$

$$= \sum_{n=0}^{\infty}\frac{(2(x_1+x_2+\ldots+x_k)t)^n}{n!}\sum_{s=0}^{\infty}\frac{(-kt^2)^s}{s!}$$

$$= \sum_{n=0}^{\infty}\sum_{s=0}^{\lfloor n/2\rfloor}\frac{(-k)^s(2k(x_1+x_2+\ldots+x_k))^{n-2s}}{s!\,(n-2s)!}t^n. \quad (2.5)$$

On the other hand, we get

$$\exp\left(2k(x_1+x_2+\ldots+x_k)t - kt^2\right) = \left[\exp\left(2(x_1+x_2+\ldots+x_k)t - t^2\right)\right]^k$$

$$= \sum_{n=0}^{\infty}\left[\sum_{n_1+n_2+\ldots+n_k=n}\frac{H_{n_1}(x_1)H_{n_2}(x_2)\ldots H_{n_k}(x_k)}{n_1!\,n_2!\ldots n_k!}\right]t^n. \quad (2.6)$$

By combining (2.5) and (2.6), one gets (2.4). □

Theorem 2.3. *For any positive integer* k,

$$\sum_{s=0}^{[\frac{1}{2}n]} \frac{(-1)^s (k)_{n-s} (2x)^{n-2s}}{s!\,(n-2s)!} = \sum_{n_1+n_2+\ldots+n_k=n} U_{n_1}(x)\,U_{n_2}(x)\ldots U_{n_k}(x). \quad (2.7)$$

Proof. Using the power series of $(1-2xt+t^2)^{-k}$, and making the necessary arrangements gives

$$\begin{aligned}(1-2xt+t^2)^{-k} &= \sum_{n=0}^{\infty} \frac{(k)_n}{n!} (2xt-t^2)^n \\ &= \sum_{n=0}^{\infty}\sum_{s=0}^{n} \frac{(-1)^s (k)_n}{s!\,(n-s)!} (2x)^{n-s} t^{n+s} \\ &= \sum_{n=0}^{\infty}\sum_{s=0}^{\lfloor n/2 \rfloor} \frac{(-1)^s (k)_{n-s}}{s!\,(n-2s)!} (2x)^{n-2s} t^n. \quad (2.8)\end{aligned}$$

In addition to this, we can write

$$\begin{aligned}(1-2xt+t^2)^{-k} &= \left((1-2xt+t^2)^{-1}\right)^k = \left(\sum_{n=0}^{\infty} U_n(x) t^n\right)^k \\ &= \sum_{n=0}^{\infty}\left(\sum_{n_1+n_2+\ldots+n_k=n} U_{n_1}(x)\,U_{n_2}(x)\ldots U_{n_k}(x)\right) t^n. \quad (2.9)\end{aligned}$$

By (2.8) and (2.9), the Equation (2.7) follows directly. □

3. Hermite-Chebyshev Polynomials

In this section, we can define the Hermite-Chebyshev polynomials by

$${}_H U_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k (n-k)!\, 2^{n-2k} H_{n-2k}(x)}{k!\,(n-2k)!}. \quad (3.1)$$

It is clear that

$$ {}_HU_{-1}(x) = 0, \quad {}_HU_0(x) = 1, \quad {}_HU_1(x) = 4x, $$

$$ {}_HU_n(-x) = (-1)^n {}_HU_n(x), $$

and

$$ {}_HU_{2n}(0) = \sum_{k=0}^{n} \frac{(-1)^n 2^{2(n-k)}}{(n-k)!\,k!}, \quad {}_HU_{2n+1}(0) = 0. $$

In view of (3.1), we consider the series

$$ \sum_{n=0}^{\infty} {}_HU_n(x)t^n = \sum_{n=0}^{\infty} \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k (n-k)!\,(2)^{n-2k} H_{n-2k}(x)}{k!\,(n-2k)!} t^n $$

$$ = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^k (2)^n (1)_{n+k} H_n(x)}{k!\,n!} t^{n+2k} $$

$$ = (1+t^2)^{-1} \sum_{n=0}^{\infty} \left(\frac{2t}{1+t^2} \right)^n H_n(x). \tag{3.2} $$

By using (1.5) and (3.2), we obtain a generating function for Hermite-Chebyshev polynomials in the form

$$ \sum_{n=0}^{\infty} {}_HU_n(x)t^n = (1-4xt+t^2)^{-1} {}_2F_0\left(\frac{1}{2}, 1; -; -\frac{16t^2}{(1-4xt+t^2)^2} \right). \tag{3.3} $$

Moreover, another generating function for Hermite-Chebyshev polynomials is given in the form

$$ \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{1}{(n-k)!} {}_HU_k(x) t^{n-k} u^n = \exp(ut)(1-4xu+u^2)^{-1} $$

$$ \times {}_2F_0\left(\frac{1}{2}, 1; -; -\frac{16u^2}{(1-4xu+u^2)^2} \right). \tag{3.4} $$

The following theorem presents a representation for the Hermite-Chebyshev polynomials and reduces to the operational rule.

Theorem 3.1. *The Hermite-Chebyshev polynomials satisfy the following representation*:

$$ {}_HU_n(x) = \exp\left(-\frac{1}{4}\frac{d^2}{dx^2}\right)U_n(2x). \tag{3.5}$$

Proof. From (3.1) and (1.6), one gets

$$ {}_HU_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k 2^{n-2k}(n-k)!\, H_{n-2k}(x)}{k!\,(n-2k)!}$$

$$ = \exp\left(-\frac{1}{4}\frac{d^2}{dx^2}\right)U(2x). \qquad \square$$

Note that the inverse of the operational (3.5) is

$$U_n(2x) = \exp\left(\frac{1}{4}\frac{d^2}{dx^2}\right){}_HU_n(x). \tag{3.6}$$

It is worth noting that, for $x = \frac{x}{2}$, the expressions (3.5) and (3.6) give another representation for the Hermite-Chebyshev polynomials in the form

$$ {}_HU_n\left(\frac{x}{2}\right) = \exp\left(-\frac{d^2}{dx^2}\right)U_n(x),$$

and

$$U_n(x) = \exp\left(\frac{d^2}{dx^2}\right)\ {}_HU_n\left(\frac{x}{2}\right).$$

Now, we can see that the integral form of Hermite-Chebyshev polynomials with on their properties and prove the following:

Theorem 3.2. *The Hermite-Chebyshev polynomials satisfy the following relations*:

$$ {}_HU_n(x) = \frac{1}{\sqrt{\pi}} \exp\left(x^2\right)\int_{-\infty}^{\infty} \exp\left(-t^2 + 2ixt\right)U_n(-2it)dt. \tag{3.7}$$

Proof. By using (1.8) and (3.1), the integral form (3.7) follows directly. □

Now, we are devoted to operational identities to the theory of exponential operators, may significantly simplify the study of Hermite-Chebyshev generating functions and the discovery of new relations.

To give examples of how the method works, we consider the sum

$$F(x, t) = \sum_{n=0}^{\infty} {}_HU_n(x)t^n, \tag{3.8}$$

and using the Equation (1) of [8], we get

$$\exp\left(-\frac{1}{4}\frac{d^2}{dx^2}\right)U_n(2x) = U_n(2x - \frac{d}{dx}). \tag{3.9}$$

By multiplying the left-hand side of (3.9) by t^n with using (1.10), we find

$$\sum_{n=0}^{\infty} U_n(2x - \frac{d}{dx})t^n = \left(1 - 2t(2x - \frac{d}{dx}) + t^2\right)^{-1}, \tag{3.10}$$

allows us to write (3.8) as

$$F(x, t) = \sum_{n=0}^{\infty} {}_HU_n(x)t^n = \frac{1}{1 - 2t(2x - \frac{d}{dx}) + t^2}$$

$$= \int_0^{\infty} \exp(-s(1 + t^2)) \exp\left(2st(2x - \frac{d}{dx})\right)ds. \tag{3.11}$$

We can therefore use the decoupling rule of the exponential is defined by

$$e^{\hat{A}+\hat{B}} = e^{\hat{A}}e^{\hat{B}}e^{-\frac{k}{2}}, \tag{3.12}$$

which holds in the hypothesis that the operators $\hat{A}$ and $\hat{B}$ satisfy the commutation brackets

$$[\hat{A},\, \hat{B}] = k, \quad [\hat{A},\, k] = [\hat{B},\, k] = 0, \tag{3.13}$$

when $[\hat{A},\, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A} = k$ [8]. By using (3.12) and (3.13), we can write the integral on the right-hand side of (3.11) as

$$\int_0^\infty \exp(-s(1+t^2))\exp\left(2st(2x - \frac{d}{dx})\right)ds$$

$$= \int_0^\infty \exp(-s(1+t^2))\exp(-(2st)^2)\exp(4sxt)\exp(-2st\frac{d}{dx})ds. \tag{3.14}$$

The second example to illustrate the usefulness of the above procedure. According to (3.6), we can write

$${}_H U_{2n}(\sqrt{x}) = \exp\left(-\frac{1}{4}\frac{d^2}{d\sqrt{x}^2}\right)U_{2n}(\sqrt{2x})$$

$$= \exp\left(-\frac{1}{2}\frac{d}{dx} - x\frac{d^2}{dx^2}\right)U_{2n}(\sqrt{2x}). \tag{3.15}$$

We now decompose the exponential operator on the right-hand side of Equation (3.15) by means of the following operational rule:

$$e^{\hat{A}+\hat{B}} = (1 + m\hat{A})^{\frac{1}{m}} e^{\hat{B}}, \tag{3.16}$$

which holds if $[\hat{A},\, \hat{B}] = m\hat{A}^2,$ we can write (3.14) as follows:

$${}_H U_{2n}(\sqrt{x}) = (1 - \frac{d}{dx})^{\frac{1}{2}} \exp\left(-x\frac{d^2}{dx^2}\right)U_{2n}(\sqrt{2x}). \tag{3.17}$$

4. Generalized Hermite-Chebyshev Polynomials

Here, we consider the generalized Hermite-Chebyshev polynomials ${}_HU_n^m(x)$ in the form

$$ {}_HU_n^m(x) = \sum_{k=0}^{\lfloor n/m \rfloor} \frac{(-1)^k (n-(m-1)k)! 2^{n-mk} H_{n-mk}^m(x)}{k!\,(n-mk)!}. \tag{4.1}$$

Consider the series

$$\sum_{n=0}^{\infty} {}_HU_n^m(x) t^n = \sum_{n=0}^{\infty} \sum_{k=0}^{\lfloor n/m \rfloor} \frac{(-1)^k (n-(m-1)k)! 2^{n-mk} H_{n-mk}^m(x)}{k!\,(n-mk)!} t^n$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^k 2^n (1)_{n+k} H_n^m(x)}{k!\,n!} t^{n+mk}$$

$$= (1+t^m)^{-1} \sum_{n=0}^{\infty} \left(\frac{2t}{1+t^m}\right)^n H_n^m(x). \tag{4.2}$$

Using (1.5) and (4.2), we obtain an explicit representation for the generating function of Hermite-Chebyshev polynomials in the form

$$\sum_{n=0}^{\infty} {}_HU_n^m(x) t^n = (1-4xt+t^m)^{-1} {}_2F_0\left(\frac{1}{2}, 1; -; -\frac{16t^2}{(1-4xt+t^m)^2}\right). \tag{4.3}$$

Finally, the above relations will be used, along with the generalized Hermite-Chebyshev polynomials can be shown to satisfy the property, to derive new properties of the family generated function by (4.3) yields as given in the following paper. It goes by itself that we can introduce the Hermite-Chebyshev polynomials

$$ {}_HU_n(x_1+x_2+\ldots+x_k) = \sum_{s=0}^{\lfloor n/2 \rfloor} \frac{(-1)^s (n-s)! 2^{n-2s} H_{n-2s}(x_1+x_1+\ldots+x_k)}{s!\,(n-2s)!}, \tag{4.4}$$

and

$$ {}_H U_n(x_1, x_2, \ldots, x_k) = \sum_{s=0}^{\lfloor n/2 \rfloor} \frac{(-1)^s (n-s)! 2^{n-2s} H_{n-2s}(x_1, x_1, \ldots, x_k)}{s!(n-2s)!}; \quad k \in \mathbb{N}. \tag{4.5} $$

The Hermite-Chebyshev polynomials of two, three index and in turn p-index in terms of series are represented as follows:

$$ {}_H U_{n,m}(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \sum_{s=0}^{\lfloor m/2 \rfloor} \frac{(-1)^{k+s} (n-k)!\,(m-s)! 2^{n+m-2k-2s} H_{n+m-2k-2s}(x)}{k!\,s!\,(n-2k)!\,(m-2s)!}, \tag{4.6} $$

$$ {}_H U_{n,m,p}(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \sum_{s=0}^{\lfloor m/2 \rfloor} \sum_{u=0}^{\lfloor p/2 \rfloor} \frac{(-1)^{k+s+p} (n-k)!\,(m-s)!\,(p-u)! 2^{n+m+p-2k-2s-2u}}{k!\,s!\,u!\,(n-2k)!\,(m-2s)!\,(p-2u)!} \times H_{n+m+p-2k-2s-2u}(x), \tag{4.7} $$

and

$$ {}_H U_{n_1, n_2, \ldots, n_p}(x) = \prod_{i=1}^{p} \sum_{k_i=0}^{\lfloor n_i/2 \rfloor} \frac{(-1)^{\sum_{i=1}^{p} k_i} \prod_{i=1}^{p} (n_i - k_i)! 2^{\sum_{i=1}^{p} n_i - 2\sum_{i=1}^{p} k_i}}{\prod_{i=1}^{p} k_i! \prod_{i=1}^{p} (n_i - 2k_i)!} \times H_{\sum_{i=1}^{p} n_i - 2\sum_{i=1}^{p} k_i}(x). \tag{4.8} $$

Further examples proving the usefulness of the present method can be easily worked out, but are not reported here for conciseness. Further applications will be discussed in a forthcoming paper.

5. Concluding Remark

One can use the same class of integral representation and operational methods for some other polynomials of several variables. Hence, new results and further applications can be obtained.

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